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PATTERN RECOGNITION BY THE METHOD OF ISOCLINES:

I. A MATHEMATICAL MODEL FOR THE VISUAL INTEGRATIVE PROCESS

by

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PATTERN RECOGNITION BY THE METHOD OF ISOCLINES: I. A. MATHEMATICAL

MODEL FOR THE VISUAL INTEGRATIVE PROCESS

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Abstract

A theoretical model for the visual integrative process is advanced
that is based upon microelectrode studies of the animal visual cortex
by Hubel and Wiesel, Jung, and others. In this theoretical model the
cortical columns of direction-sensing elements found by Hubel and Weisel
are interpreted as sets of isoclines making up the visual field. The
form of a visual image corresponds to a family of contours ("orbits" or
"paths", in the terminology of differential equations) that pass through
those isoclines stimulated by the corresponding retinal regions. A con-
tour of an image is thus approximated in the visual cortex as an iso-
clinic polygonal arc.

Hubel and Wiesel's microelectrode studies are thus strongly sug-
gestive of a differential equation interpretation of the visual inte-
grative process. The invariance of the visual integrative process under
rigid motions (translation and rotation) and perspective that is required
by the psychological phenomenon of Gestalt enables us to specify the
particular form of the differential equations involved. The latter
then permit a ready explanation of such visual phenomena as McKay's
complementary after images, the whirling spiral images evoked under
flicker, and the alpha rhythm and its desynchronization.

I.   INTRODUCTION

Historically, pattern recognition has been synonymous with statis-
tical classification. A pattern is abstracted as a point in a finite-
dimensional sample space, the dimensionality of which corresponds to
the number of identifiable characteristics of the pattern. The procedure
for pattern recognition then involves partitioning the n-dimensional
sample space in such a way that the sample points representing one class
of patterns are essentially separable from the region associated with
any other class. As such, this is a generalized form of the statistical
problem of classifying an observation into one of finitely many popula-
tions (Anderson, 1958, Ch.6; Rao, 1952, Chs. 8,9; Sebestyen, 1962;
Abramson, 1963; Braverman, 1963).

Certain aspects of this approach run counter to what one ordinarily
thinks of as constituting a "pattern", namely, some sort of connected
configuration without essential discontinuities in its geometry, smooth-
ness, symmetry or the like. An optical image, for example, may be
thought of as a collection of contours (or paths). It is not clear
that the optimal recognition procedure lies in abstracting this image
to a single point of a higher-dimensional space obtained from a sub-
division of the image into n rectangular elements. In fact Swerling
(1962) has given arguments indicating that analysis of a random surface
in terms of contours requires appreciably less information, and similar
economies have been noted by Attneave and Arnoult (1956), Capon (1959),
Unger (1959), Wholey (1961), and others.

For such patterns as optical (or visual) images it seems clear that the characterization would be improved by retaining their configurational aspects. Since an image may be characterized by the arrangement of contours in it, the usual finite-dimensional characterization of classical pattern recognition theory should be improved by taking the full geometry into account. A number of approaches are then possible.

Each contour in the figure can be considered as a point in an infinite-dimensional space, namely the semi-Banach space of curves (Fréchet, 1962). This approach may be particularly well suited to a generalization of the classical pattern recognition method of separating regions corresponding to the various classes by means of finite-dimensional hyperplanes. The question of linear separation (classification by a separating hyperplane) of subspaces of reflexive Banach spaces has recently been settled by James (1964).

However, the less abstract approach embodied in contour representation by means of tricoordinates (abscissa, ordinate, and slope) (Attneave and Arnoult, 1956) apparently has more far-reaching implications for such pattern recognition phenomena as the visual integrative process. Our study of pattern recognition will therefore be couched largely in terms of tricoordinates, or direction-field elements.

Still another approach, that in fact is closely related to that of tricoordinates (especially in the recognition aspect itself), follows from regarding visual patterns as Markov processes in time and space.

The work of Beutler (1963) leads naturally to such in interpretation, once the statistical decision problem arising out of the association between differential equation and contour comes under consideration. We note that trajectories are involved in both approaches.

As noted above, the tricoordinates of a point on a curve consist of the rectangular coordinates of the point and the local value of the derivative

$$(x, y, \frac{dy}{dx}). \tag{1}$$

A well known result from the theory of differential equations (Bieber-bach, 1956, p. 27) states that an approximative polygonal arc made up of n such elements, each of which consists of a point and the associated local tangent, can be made to approach arbitrarily close to an integral curve of the differential equation

$$\frac{dy}{dx} = f(x, y), \tag{2}$$

so long as f remains continuous. The essential feature of our theoretical model for the pattern recognition process is the approximation of a contour by such a polygonal arc as that described above. Form vision rather than brightness is of primary interest in the present context, and color vision is not considered at all. Brightness is in principle readily taken into account by including in the form character specified by f(x,y) in (2) the frequency of firing of the particular neurons involved (Rushton, 1961; Jung, 1961 a,b).

As will be shown below, the polygonal arc approximation to a shape leads naturally, in the context of the visual integrative process, to the graphical integration technique known as the method of isoclines (Andronow and Chaikin, 1949; Cunningham, 1958; Kaplan, 1958). The Gestalt character of visual pattern recognition demands invariances of a type associated with Lie groups (Cohn, 1957; Ince, 1956; Kowalewski, 1950; Lie and Scheffers, 1893; Yano, 1957). Invariance under the appropriate Lie groups tells us what particular form the differential equations must have. The theory of these differential equations is then developed and related to such visual phenomena as Jung's A,B,C,D,E type neurons (Jung, 1961 a,b), MacKay's complementary after images, the alpha rhythm and its desynchronization, and the whirling spirals often evoked under flicker (Walter, 1953). Sign reversal in this system of differential equations governing the visual integrative process is apparently closely related to such conditions as developmental dyslexia (Bender, 1957; Money, 1961; Stuart, 1963), but a complete discussion of this will be reserved for a separate publication. The statistical decision theory aspects of pattern recognition via the method of isoclines will also be only briefly taken up in the present paper.

II. MICROELECTRODE EXPLORATION OF THE VISUAL CORTEX

The use of microelectrodes for deep penetrations of the animal cortex in the manner of Mountcastle (Mountcastle, 1957; Powell and Mountcastle, 1959) has provided neurophysiologists with a powerful new tool for determining the response characteristics of single cortical neurons. Microelectrode studies have been made of the cat auditory pathway (see Katsuki, 1961, and the references cited there); the somatic afferent system in cat and monkey (Mountcastle, 1961); and the visual pathway in the cat (Hubel, 1963a; Jung, 1961 a,b). It is primarily the latter that will concern us here.

1. Hubel and Wiesel's Microelectrode Studies of the Animal Visual Cortex

In their series of microelectrode studies of the animal visual pathway Hubel and Wiesel (Hubel and Wiesel, 1962; Hubel, 1963 a,b) mapped out the microresponse fields in the brain under stimulation of the retina by very small patterns of light. The visual stimulus is initially processed by the retinal cells in the vicinity of the illuminated spot. In this processing the phenomenon of lateral inhibition plays a prominent role. The visual stimulus is then transmitted along the optic nerve and through the lateral geniculate body to the striate (or visual) cortex. The cells in the retina and the lateral geniculate body have a predominantly circular response (fig.1) of either excitatory or inhibitory type. The shape of the response surface looks very similar to that for a single cable in a multi-channel telephone cable, where very small sideband levels are used to reduce crosstalk.
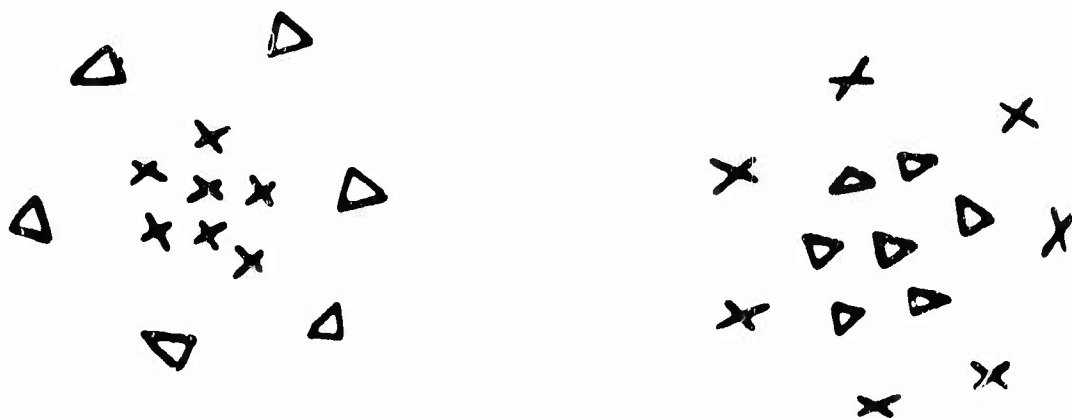
Fig. 1 Circular response of receptive fields of the lateral
geniculate body. (x, areas giving excitatory -"on"-
responses; Δ, areas giving inhibitory -"off"- responses)
From Hubel and Wiesel (1962).

The cells in the striate cortex on the other hand respond in an

entirely different manner, being arranged in fields with highly linear

responses (fig. 2). One type of such fields, termed "simple", responds

only when the linear light stimulus encounters the boundary between its

inhibitory and excitatory sub-fields. The response is strongly direc-

tional, an angular difference of at most 10° (and usually considerably

less) between the axes of linear light source and receptive field being

enough to annul the response.

The second type of receptive field, termed "complex" by Hubel and

Wiesel, is also highly orientation sensitive, but is characterized by

exhibiting a sustained response as the light stimulus is moved over

relatively large regions of the retina. The complex fields behave

as if they received their afferents from a large number of the cortical

cells in simple fields, all of which have essentially the same axis

orientation but which vary slightly among themselves in retinoptic pro-
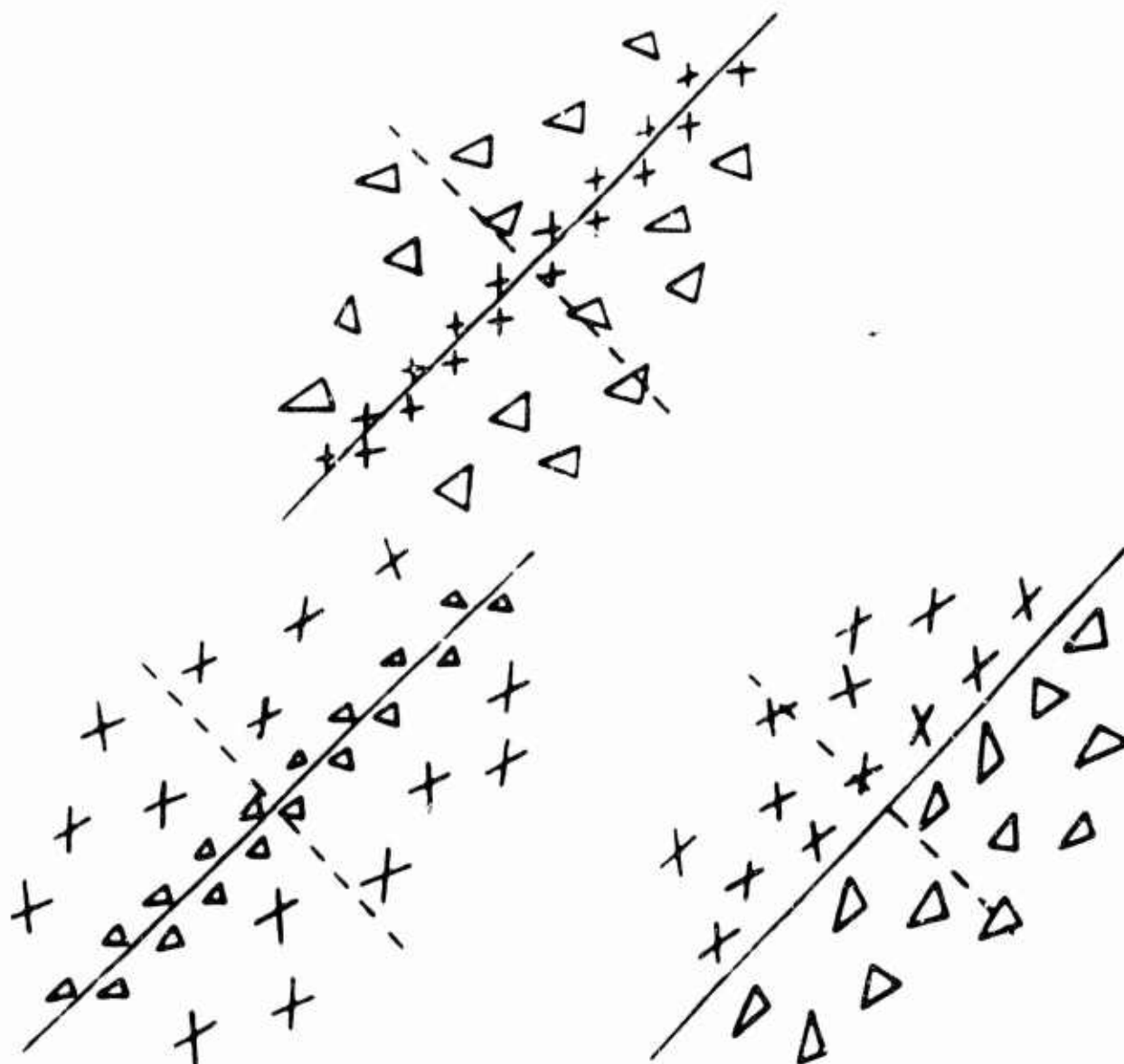
jection.

Fig. 2. Linear response of receptive fields of the visual
cortex. (x, areas giving excitatory -"on"- responses;
Δ, areas giving inhibitory -"off"- responses. Receptive
field axes are indicated by continuous lines through
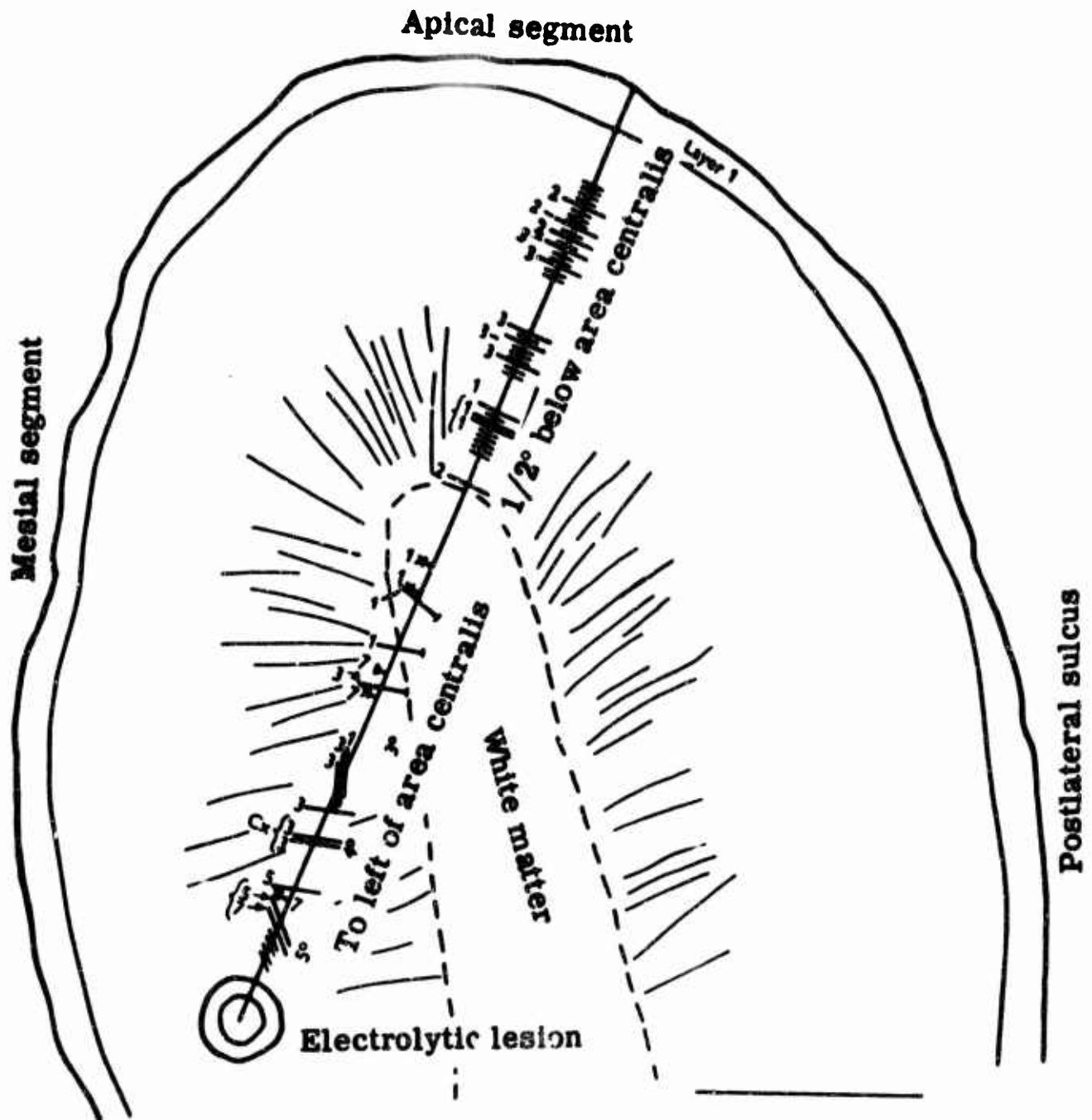field centers.) From Hubel and Wiesel (1962).

Fig. 3. Reconstruction of microelectrode penetration through the lateral gyrus (from Hubel and Wiesel (1962)). The electrode entered the apical segment normal to the surface and advanced parallel to the deep fiber bundles (indicated by radial lines) until white matter was encountered. In the further advance of the electrode through the gray matter of the mesial segment the course was oblique. The receptive field orientations of the cortical cells are indicated by longer lines transverse to the electrode track; crossbars at the right hand end of the lines indicate axons. Approximate positions of receptive fields on the retina are shown to the right of the penetration. Short lines transverse to the electrode track indicate unresolved background activity.

Receptive field axes* of all orientations are found in the visual cortex, the only apparent regularity consisting of alignment either nearly parallel or nearly perpendicular to the radial fiber in the immediate vicinity. This feature is evident from figs. 3,4,5, reproduced here from Hubel and Wiesel (1962) and Hubel (1963b).
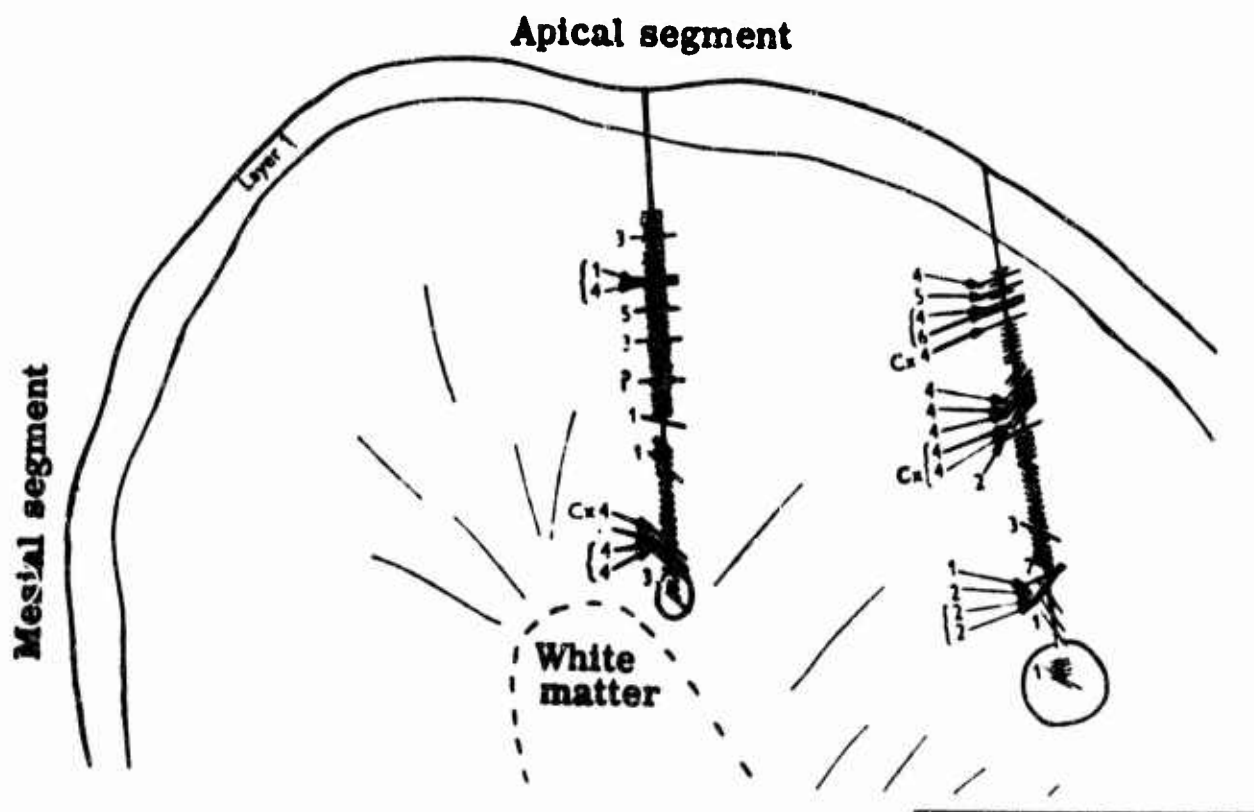

Fig. 4. Reconstruction of two penetrations in the apical segment of the post-lateral gyrus, near its anterior end (from Hubel and Wiesel (1962)). Legend as for fig. 3. Note the abrupt transition from receptive fields parallel to the radial fibers to transverse receptive fields in the terminal portion of the rightmost track.

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* The receptive field axis is a hypothetical line through the center of the receptive field and directed along the boundary between excitatory and inhibitory regions (fig. 2).

The principal characteristic that distinguishes complex fields from simple fields is that the former respond with sustained firing to moving lineal stimuli. The simple fields, responsive only at the boundary between excitatory and inhibitory regions (fig. 2), fire on the other hand only for the relatively brief period when the moving stimulus crosses the boundary. It would thus appear that the relatively great sensitivity of the visual cortex to moving stimuli is intimately bound up with the complex fields.

The cells themselves, simple and complex, are apparently arranged in cylindrical columns perpendicular to the local cortical surface. The column axes seem to be essentially the same as those of the deep fiber bundles. Within these columns, the receptive field axes appear to be conjugate (that is, each other's orthogonal trajectories) (figs. 3 and 5), some axes being (essentially) perpendicular to the local deep fiber bundle, the others (essentially) parallel to it.

The projection of retinal regions upon the visual cortex is ordered, i.e., the receptive fields contained in a given cortical column respond to corresponding portions of the two retinas. Hubel and Wiesel (1962) state that within such a column defined by common orientation of receptive field axes there is no apparent progression in field positions along the retina as the electrode advances, and conclude that at the microscopic level the retinoptic representation, within a given cortical column, no longer strictly holds. Hubel and Wiesel do detect an apparently random staggering of receptive field
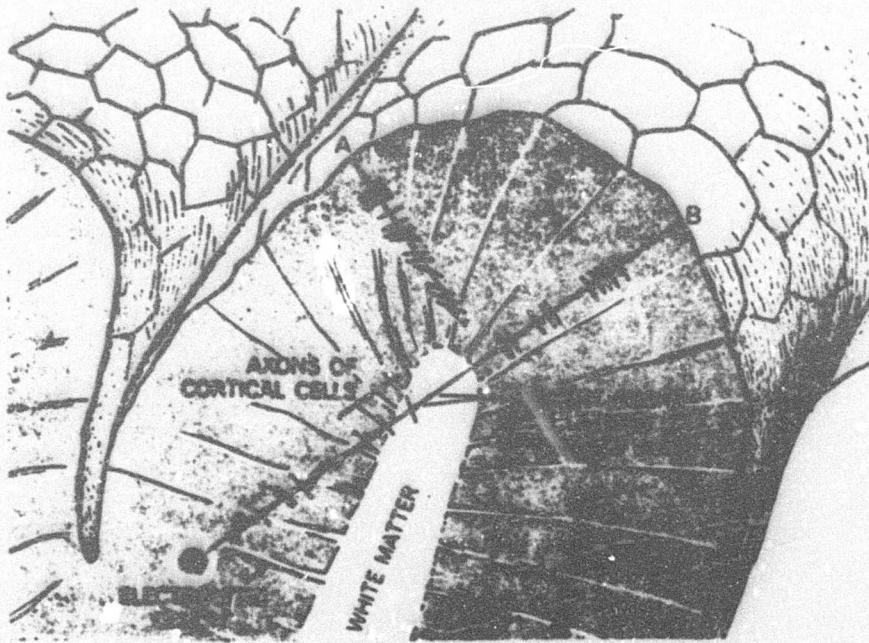
Fig 5. Diagrammatic representation of the cortical columns of
receptive fields (after Hubel (1963b)). Lines A and B
indicate tracks of two microelectrode penetrations.

positions for a given cortical column. Upon close examination this
staggered structure does, however, seem to display a systematic varia-
tion of retinoptic position, upon which is superimposed a random "jitter".
An ordered relation between progressive depth of electrode penetration
and distance from the area centralis is detectable (fig. 6): For a
given angular bearing with respect to the area centralis, the farther
the retinal receptive field is from the area centralis, the greater,
in general, the depth of the associated cortical receptive field below
the cortical surface.

The degree to which the observed random variation represents true
random staggering of the cytostructure is not clear. It would seem
that the effects of deformation of the cytostructure by the passage of
the electrode, and angular errors of electrode direction with respect
to column axis must enter as well as the randomness present in the
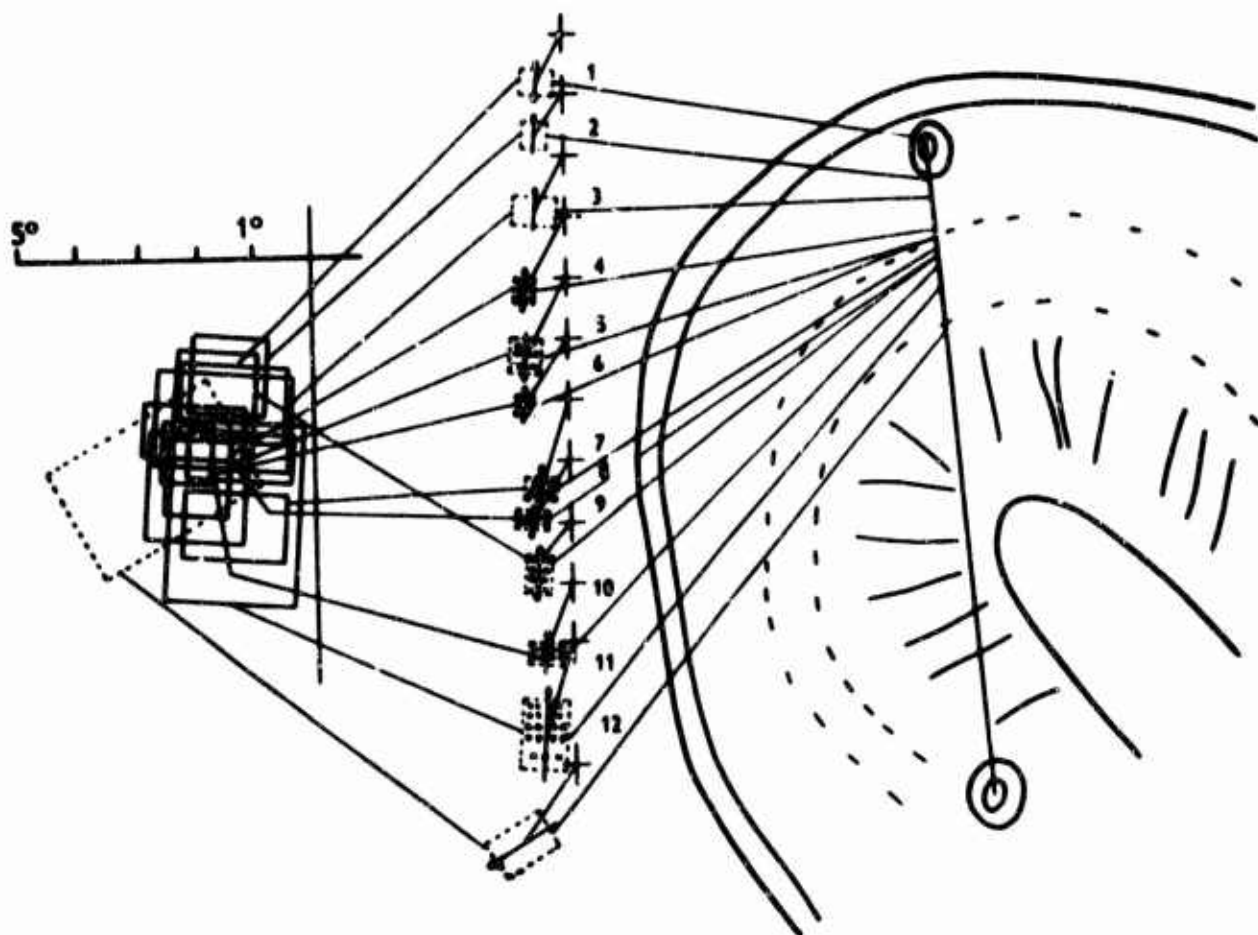cytoarchitecture itself.

Fig. 6. Reconstruction of portion of electrode track through apical and
mesial segments of post-lateral gyrus near its anterior end made
during multiple recordings (after Hubel and Wiesel (1962)).
(Only the first twelve cells are represented. Broken curves
indicate the boundaries of layer 4.) The overlapping rectangles
in the leftmost part of the figure display the superposition of
receptive fields upon the retina, whose coordinate system appears
directly above (origin referred to area centralis).

The progression of similarly-oriented receptive fields is shown
in the central portion of the figure. The rectangles outlined
with broken lines are the receptive fields. The crosses indi-
cate the relative position and bearing of the corresponding
area centralis. The numbers to the right give the field sequence:
1-12. The field following field 12 is apparently transverse to
an adjacent radial fiber. The arrows show the preferred direction
of movement of a slit oriented parallel to the receptive field
axis.

The resemblance of these cortical columns of oriented direction fields to the method of isoclines (or "direction field method") for the graphical solution of differential equations (Andronow and Chaikin, 1949; Cunningham, 1958; Kaplan, 1958) is indeed striking (fig. 7). Regarding an optical image as an ensemble of contours, we hypothesize the following theoretical model for the visual integrative process:

> The whole visual field is permanently filled up with tiny directional elements that are essentially isoclines (elements of same slope) arranged in cortical columns. A visual image stimulates only those isocline nets that correspond to forms in the image. The ensemble of slope elements corresponding to a given image contour thus comprises a polygonal arc approximation to the solution of the differential equation defining that image contour.

The significant feature is that, in terms of what actually happens in the visual cortex, the image is characterized not so much by the forms that make it up but rather by the differential equations (and initial or boundary conditions) that govern these shapes. The visual cortex thus seems to possess a built-in structure for graphical integration of the differential equations of visual forms.

2. Other Microelectrode Studies of the Visual Pathway.

The many other microelectrode studies of the cerebral cortex and the visual pathway that have been made are well described by Amassian (1961) and Jung (1961, a,b). These studies have been voluminous, but

only those aspects that bear directly upon our theoretical model will be summarized here. In particular the work of Jung and his school (Jung, 1961 a,b) seems to have very direct implications for the kind of excitations appropriate to our theoretical model. In addition these papers are notable for a thoroughgoing summary of the connections between the neurophysiology of the feline visual cortex and the psychophysiology of vision in man.

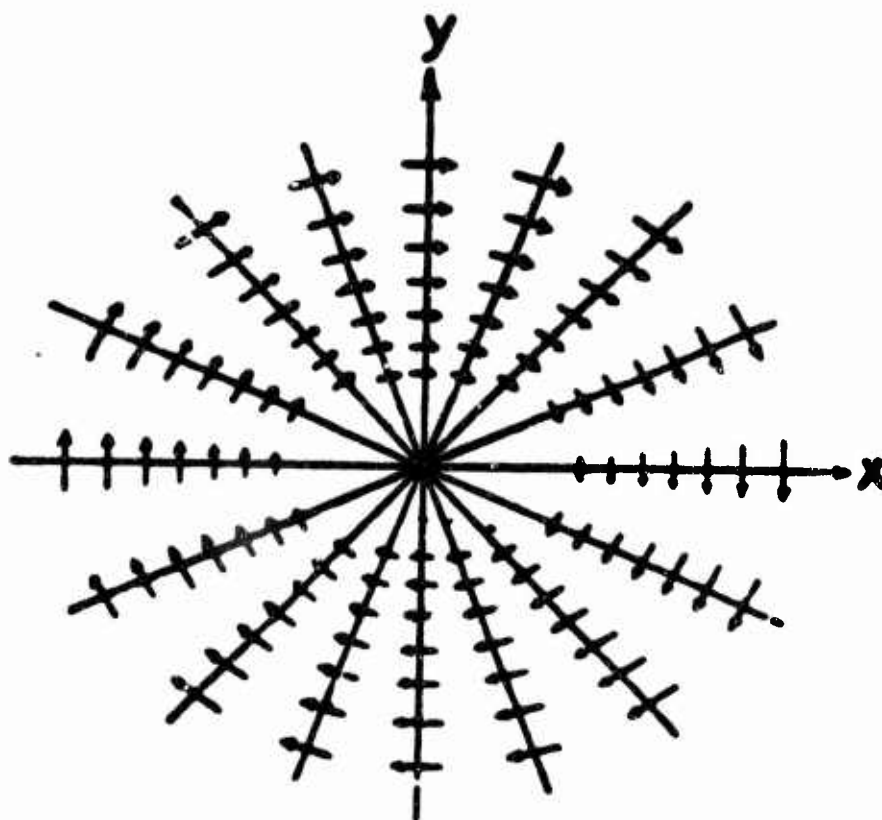Figure 7. The method of isoclines. Small arrows indicate directed slope elements. The isoclines are the radial lines, each of whose slope elements are parallel. (After Andronow and Chaikin (1949)).

Jung (1961 a,b) distinguishes five classes of cortical neurons according to their characteristic excitation-inhibition response under either diffuse or patterned light stimuli. Type A neurons are in a

class by themselves. They fire regularly at a frequency of from 8 to 15 discharges per second but show no response to either light or darkness. They are apparently driven by nonspecific reticulothalamic stimuli and constitute a "... medium background of excitation in the cortex..." (Jung, 1961a, p. 631).

Type B,C,D,E neurons on the other hand are light (and/or dark) responsive, and are classified according to their characteristic response under illumination of the retina. Type B and D neurons are apparently duals (antagonists) of each other. The type B neurons impart brightness information; they are activated by light and inhibited by darkness. The type D neurons on the other hand impart relative darkness information; they are inhibited by light and activated by darkness. Type C neurons are inhibited at both the onset and termination of a light stimulus. Type E neurons show a relatively long-delayed response to a light stimulus but a strong rapid reaction to the cessation of light. The responses of the C and E neurons appear on occasion to complement each other in somewhat the same way as the B and D neurons do (Jung, 1961 a, Fig. 3; Jung, 1961 b, Fig. 1).

Although the shapes in the visual image constitute the forcing functions for the differential equations of the visual integrative process, it is the several categories just defined that determine the microtime behavior of those cortical neurons stimulated by the image. It is this latter aspect of the neuronal behavior that will be needed for relating our theoretical model to such phenomena as the alpha rhythm and its desynchronization.

III. THE METHOD OF ISOCLINES AND ITS CONNECTION WITH THE VISUAL INTEGRATIVE PROCESS

In the graphical integration method using isoclines successive numerical values of the quantity m are first computed according to the equation

$$m = f(x,y), \qquad (3)$$

where $f(x,y)$ is given by (2). The value of m thus determined for the point (x,y) is then assigned to the slope at that point. A curve connecting all points at which the slope has the same value is called an isocline. The effect of this procedure is to reduce the differential equation (2) to a sequence of equivalent algebraic equations of the form (3), which then specify the locus of those values of x and y along which the integral curves of (2) have a particular slope (fig. 7).

Thus, once an isocline is plotted in the x,y-plane, one can pick out a set of line elements along the isocline that have the prescribed slope m. The value of m is then changed and another isocline curve plotted, and so on. The result is to eventually fill up the entire x,y-plane with isoclines, each carrying directed line elements of a given slope (fig. 7).

Next one starts from a given initial point and sketches a smooth curve, always following the slope of successive line elements in adjacent isoclines. This procedure yields a graphical solution of the initial value problem. A different initial value will in general lead to a different graphical solution. The process is in principle capable of

any prescribed degree of accuracy, depending upon the scale, the distance between adjacent isoclines, and the total number of slope elements along each isocline.

It thus appears from Hubel and Wiesel's microelectrode studies (Hubel and Wiesel, 1962) that the visual cortex carries a direct counterpart of an isocline net, the slope elements being Hubel and Wiesel's simple fields and the isoclines the cortical columns. The shapes in an object imaged on the retina are identified as the envelopes of the corresponding cortical fields that are stimulated. Tracing through the isocline net that is stimulated is the analogue of the sketching process described in the preceding paragraph.

Such a model for the visual integrative process has many points of contact with the psychophysiology of the visual integrative process. It is well grounded on neurophysiological realities. It leads at once to explanations of such phenomena as the visual images evoked under flicker, MacKay's complementary after images (MacKay, 1961) the alpha rhythm and its desynchronization, the specific language disability known as developmental dyslexia (Bender, 1957), and the visual Gestalt (de Hirsch, 1962). The latter appears in its simplest aspects as the invariance of the differential equations of the visual integrative process under the appropriate Lie transformation groups. Memory and the more complex aspects of the visual Gestalt, however, apparently require statistical decision and prediction theory for the most general sort of concepts in Lie groups and differential topology (Auslander and Mackenzie, 1963; Munkres, 1963; Steenrod, 1951; Yano, 1957).

IV. THE DIFFERENTIAL EQUATIONS OF THE VISUAL INTEGRATIVE PROCESS

The isocline interpretation of Hubel and Wiesel's microelectrode studies implies that the visual integrative process is based upon some first order differential equation (or equations). The question then arises - which first order differential equations? The answer is provided by the Gestalt requirement that the visual image be invariant, i.e., still recognizable as an entity, under rotations, translations, and perspective transformations (magnifications, or dilatations). We therefore seek those differential equations whose family of solutions is invariant under the combined group of rigid motions (translations and rotations) and magnifications. The determination of these invariant differential equations thus reduces the problem to an elementary computation in the theory of Lie groups.

That the features of the visual Gestalt (de Hirsch, 1962) must involve invariance under a group of transformations has been recognized by several investigators (Pitts and McCulloch, 1947; Culbertson, 1957; Rashevsky, 1960). Pitts and McCulloch viewed a pattern of stimuli as the arithmetic mean of a functional over all the transformations of the finite group involved. Culbertson considered possible neuronal networks that would generate, in the limit of translations, stimulus patterns invariant under translations, rotations, and magnifications. Rashevsky's approach was based on an interaction of several central complexes of neuroelements in such a way as to produce the desired invariance at a higher, coordinating center. However, none of these

studies introduced the Lie group aspect, and it is believed that this
is the essential new feature that provides the key to the actual workings
of the visual integrative process.

It will be assumed that the Lie groups and differential equations
involved are 2-dimensional. An optical image on the retina is a 2-dimen-
sional mapping of the 3-dimensional world, and although nonplanar, can
be thought of as a 2-dimensional manifold*. As Davson (1949) puts it,
"The retina has ... become specialized by the development of apparently
vertical and horizontal meridians; these act as a pair of rectangular
coordinates through the fovea to which the position of any point or line
is referred."

1. Resume of Essentials of Lie Group Theory

We shall here review briefly only those aspects of Lie group theory
that will be needed in the application to Gestalt psychology made below.
The reader will be assumed to be acquainted with Ch. IV of (Ince, 1956)
or its equivalent. A complete treatment of the classical projective
group may be found in (Lie and Scheffers, 1893) or (Kowalewski, 1950).
A reader interested in a modern abstract treatment of Lie groups is
referred to the monographs by Cohn (1957) or Nomizu (1956), or to
(Auslander and Mackenzie, 1963).

The transformation in the plane

$$(x_1, y_1) = T_\sigma(x, y) = (g(x, y), h(x, y)) \tag{4}$$

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* A manifold is a connected compact topological space that is locally
Euclidean.

constitut, a group if it has the usual group properties (closure, existence of identity and inverse, associativity). The subscript σ denotes a parameter (which is also understood to be in the arguments of g and h). When $T_{\sigma_2} T_{\sigma_1}(x,y) = T_{\sigma_2 + \sigma_1}(x,y)$, the family (4) of transformations is said to form a <u>one-parameter continuous group</u>, denoted by G_1. A Lie group is a continuous group on a differentiable manifold suc.. that the differentiability properties are consistent with the group properties.

The differentiability structure enters via the infinitesimal transformation (or generator) of the group. Thus the system of differential equations equivalent to (2):

$$\frac{d}{d\sigma}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} , \qquad Q/P = f(x,y), \tag{5}$$

written in the form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix} d\sigma,$$

can be thought of as an "infinitesimal transformation"

$$(x_1, y_1) = (x + \delta x, y + \delta y) = (x + P(x,y)\delta\sigma, y + Q(x,y)\delta\sigma). \tag{6}$$

Each transformation of the group can be obtained by iterating the infinitesimal transformation $(x,y) \rightarrow \rightarrow (x + \delta x, y + \delta y)$ a sufficient number of times. In this sense (6) defines the infinitesimal generator of the group.

If we consider the variation induced on an arbitrary differentiable function by the infinitesimal transformation (6) of the space variables, we obtain the following representation for the <u>infinitesimal generator</u> U of the group:

$$U = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{7}$$

In particular, $Ux = P(x,y)$, $Uy = Q(x,y)$, so that an alternate form for (7) is

$$U = (Ux)\frac{\partial}{\partial x} + (Uy)\frac{\partial}{\partial y}. \tag{8}$$

The <u>finite equations</u> of the group are given in terms of U by the Maclaurin series

$$F(x_1,y_1,\sigma) = F(x,y) + \sum_1^\infty (U^n F)\sigma^n/n! \equiv e^U F, \tag{9}$$

where U^n indicates an n-fold application of the operator (7). In particular the finite equations for the space variables themselves are

$$x_1 = x + \sum_1^\infty (U^n x)\sigma^n/n!, \qquad y_1 = y + \sum_1^\infty (U^n y)\sigma^n/n!. \tag{10}$$

We turn now to the matter of <u>invariance under a Lie group</u>, and first of all define <u>invariance of a function</u> $F(x,y)$ under the action of the group. This means that if $(x,y) \rightarrow\rightarrow (x_1,y_1)$ under some operation(s) of the group, $F(x_1,y_1) = F(x,y)$ for all σ. A n.a.s.c. for invariance of a function F under a group G, whose infinitesimal generator is U, is that $UF \equiv 0$, which is, formally, a partial differential equation. Hence <u>every one parameter group G_1 in two variables</u>

has one and only one independent invariant

$$C = F(x,y),$$ (11)

obtained by solving the system

$$\frac{dx}{P(x,y)} = \frac{dy}{Q(x,y)} \,.$$ (12)

It is clear that if $P = 0 = Q$ for some point (x_0,y_0), then regardless of what $F(x,y)$ may be, $UF(x_0,y_0) = 0$, and (x_0,y_0) is a fixed point of the transformation. Such points are said to be absolute, or point, invariants.

An integral curve of the differential equation $\frac{dy}{dx} = Q/P$ is generated by successive applications of $U = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, starting from some initial point. Hence the family of integral curves (11) is invariant under the group, and is said to display curve invariance.

A family of curves may also be invariant in the sense that the individual curves, while shifting under U, still make up the same family as before. A n.a.s.c. for family invariance is that

$$UF = f(F),$$ (13)

where $f(F)$ is an ordinary function of F. Thus, for example, for the rotation group

$$x_1 = x \cos \sigma - y \sin \sigma, \qquad y_1 = x \sin \sigma + y \cos \sigma,$$ (14)

whose infinitesimal generator is

$$U_{rot.} = - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \,,$$ (15)

we have, letting $F = y/x$, that

$$UF = (y/x)^2 + 1 = F^2 + 1.$$

That is, $\int = 1 + F^2$, thus establishing the invariance, under rotation, of the family $y/x = $ const. of straight lines through the origin.

If $F_1(x,y,z) = C_1$, $F_2(x,y,z) = C_2$ are two independent solutions of the partial differential equation

$$U'F = P(x,y)\frac{\partial F}{\partial x} + Q(x,y)\frac{\partial F}{\partial y} + Z(x,y,z)\frac{\partial F}{\partial z} = 0,$$

then the most general equation invariant under the group whose infinitesimal generator is U is of the form $\Phi(F_1, F_2) = 0$, or in solved form, $F_2 = \varphi(F_1)$. This result is basic to the determination of the class of differential equations invariant under a given Lie group.

It can readily be shown that the augmented transformation

$$x_1 = g(x,y;\sigma), \quad y_1 = h(x,y;\sigma), \quad y_1' = p(x,y,y';\sigma),$$

acting on the tricoordinates (x,y,y') in the plane, forms a group. This group constitutes the _extended group_ of the given group G_1. Its infinitesimal generator is

$$u' = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} + Z(x,y,y')\frac{\partial}{\partial y'}, \tag{16}$$

where $Z(x,y,y')$ is given by the formula

$$Z(x,y,y') = \frac{\partial Q}{\partial x} + \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) y' - \frac{\partial P}{\partial y} y'^2. \tag{17}$$

We are now in a position to determine the form of those first order differential equations invariant under a given Lie group.

Suppose then that

$$F(x,y,y') = 0 \qquad (18)$$

is a general first order differential equation invariant under the extended group (16). The n.a.s.c. for this invariance is the relation

$$U'F = 0, \qquad (19)$$

which of course holds, either of itself or by (18). This is a partial differential equation whose general solution requires two particular integrals of the system

$$\frac{dx}{P(x,y)} = \frac{dy}{Q(x,y)} = \frac{dy'}{Z(x,y,y')} \ . \qquad (20)$$

If $F_1(x,y) = C_1$ is a solution of the first pair of equations, then F_1 is independent of y' since P and Q are. Now suppose $F_2(x,y,y') = C_2$ is a solution of the full system, and so necessarily involves y'. If $H(F_1)$ denotes an arbitrary function of F_1, the new function

$$F = F_2 - H(F_1)$$

satisfies the partial differential equation (19), i.e., $U'F = 0$ for this F.

The groups that we have mentioned as basic to the visual Gestalt (rotation, magnification, translation) are each subgroups of the general projective group of the plane. The finite equations of the

projective group

$$x_1 = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \qquad\qquad y_1 = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}, \qquad (21)$$

define an 8-perameter continuous group that takes straight lines into
straight lines and leaves at least one point of the plane fixed (Lie and
Scheffers, 1893). Through this fixed point there passes an invariant
line. The infinitesimal generator of the general projective group is a
linear combination of the eight operators

$$\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ x\frac{\partial}{\partial x}, \ y\frac{\partial}{\partial x}, \ x\frac{\partial}{\partial y}, \ y\frac{\partial}{\partial y}, \ x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \ xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \quad (22)$$

The most general projective transformation of the plane that leaves the
line at infinity (in visual terms, the horizon) invariant consists of a
linear combination of the first six subgroups in (22) (Lie and Scheffers,
1893).

Certain linear combinations of the subgroups (22) will be of
especial interest from the standpoint of the visual Gestalt. These
may be conveniently summarized in the following table:

Group	Infinitesimal Generator	Finite Equations
Rotation	(15)(counterclockwise) or $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ (clockwise)	(14) or $x_1 = x\cos\sigma + y\sin\sigma$, $y_1 = -x\sin\sigma + y\cos\sigma$, resp. (23)
Magnification (or similarity or perspective)	$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$x_1 = e^\sigma x, \ y_1 = e^\sigma y$ (24)
Translation	$\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}$	$x_1 = x + \sigma, \ y_1 = y + \gamma\sigma$ (25)
Rigid motions	$\varkappa_1(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) + \varkappa_2\frac{\partial}{\partial x} + \varkappa_3\frac{\partial}{\partial y}$	$x_1 = x\cos\theta - y\sin\theta + a$, (26) $y_1 = x\sin\theta + y\cos\theta + b$
Affine (or linear group)	$(a_1 + a_2 x + a_3 y)\frac{\partial}{\partial x} +$ $+ (\beta_1 + \beta_2 x + \beta_3 y)\frac{\partial}{\partial y}$	$x_1 = (a_1 x + b_1 y + c_1),$ (27) $y_1 = (a_2 x + b_2 y + c_2),$ $a_1 b_2 - a_2 b_1 \neq 0.$

2. Invariant Differential Equations of the Visual Integrative Process

Since the visual Gestalt demands that images be recognizable by their intrinsic over all character, however they be presented to the viewer (within certain limits), the similarity and rigid motion groups, (24) and (26), will be basic to our further considerations. We therefore consider the following infinitesimal generator

$$U = (-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}) + (\frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}) + \rho(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}),$$

or

$$U = (\rho x - y + 1)\frac{\partial}{\partial x} + (x + \rho y + \gamma)\frac{\partial}{\partial y}, \tag{28}$$

which combines the infinitesimal generators of rotation, translation, and magnification groups in sufficiently general form, and seek the most general differential equation invariant under the combined group of (28).

According to (17) and (20) the invariant differential equations may be determined from the system

$$\frac{dx}{\rho x - y + 1} = \frac{dy}{x + \rho y + \gamma} = \frac{dy'}{1 + y'^2}. \tag{29}$$

The change of variables

$$\rho u - v = \rho x - y + 1, \qquad u + \rho v = x + \rho y + \gamma \tag{30}$$

takes (29) into the equivalent system

$$\frac{du}{\rho u - v} = \frac{dv}{u + \rho v} = \frac{dy'}{1 + y'^2}. \tag{31}$$

According as we consider

$$z = u/v \qquad \text{or} \qquad w = v/u = 1/z, \tag{32}$$

as the dependent variable we obtain two different forms of the first pair of eqs. (31). That is,

$$\frac{dz}{dv} = -\frac{1}{v}\frac{1 + z^2}{z + \rho}, \qquad \text{or} \quad \frac{dw}{du} = \frac{1}{u}\frac{1 + w^2}{\rho - w}.$$

Separating variables, one obtains the respective equations

$$-\frac{dv}{v} = \frac{z + \rho}{1 + z^2}\,dz, \qquad \text{and} \qquad \frac{du}{u} = \frac{\rho - w}{1 + w^2}\,dw, \tag{33}$$

which may be integrated by quadratures.

Thus, the first equation has the solution

$$\mathcal{C}/v = (1 + iz)^{\frac{1}{2}(1-i\rho)}(1 - iz)^{\frac{1}{2}(1+i\rho)}$$

or

$$\mathcal{C} = (u^2 + v^2)^{\frac{1}{2}}\exp\{\arctan\tfrac{u}{v}\}. \tag{34}$$

Similarly, from the second equation of (33),

$$\mathcal{C}_\perp/u = (1 + iw)^{\frac{1}{2}(1+i\rho)}(1 - iw)^{\frac{1}{2}(1-i\rho)},$$

or

$$\mathcal{C}_\perp = (u^2 + v^2)^{\frac{1}{2}}\exp\{-\arctan\tfrac{v}{u}\} = e^{\pi/2}\mathcal{C}. \tag{35}$$

With z and w as new variables the system (31) may be written as

$$-\frac{dz}{1 + z^2} = \frac{dw}{1 + w^2} = \frac{dy'}{1 + y'^2}. \tag{36}$$

The pair consisting of the first and third of eqs. (36) has the solution

$$\arctan z + \arctan y' = \Theta(C),$$

or

$$\frac{z + y'}{1 - zy'} = \tan \Theta(C) = - \Omega(\sqrt{u^2 + v^2}\ e^{\arctan(u/v)}),$$

where Ω is an arbitrary function of the arguments shown. Solving for $y' = dv/du$, we obtain the following form for the most general differential equation admitting the group of (28):

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{-u - \Omega(\sqrt{u^2 + v^2}\ e^{\arctan(u/v)})v}{v - \Omega(\sqrt{u^2 + v^2}\ e^{\arctan(u/v)})u}. \tag{37}$$

The other pair of equations in the system (36) yields the solution

$$\arctan w - \arctan y'_{\perp} = \Theta_{\perp}(C_{\perp}).$$

This equation can be solved in a similar manner to obtain the form of the most general invariant differential equation governing the conjugate curve:

$$\frac{dy_{\perp}}{dx_{\perp}} = \frac{dv_{\perp}}{du_{\perp}} = \frac{-v_{\perp} + \Omega_{\perp}(\sqrt{u_{\perp}^2 + v_{\perp}^2}\ e^{-\arctan(v_{\perp}/u_{\perp})})u_{\perp}}{-u_{\perp} - \Omega_{\perp}(\sqrt{u_{\perp}^2 + v_{\perp}^2}\ e^{-\arctan(v_{\perp}/u_{\perp})})v_{\perp}}. \tag{38}$$

If furthermore the form of $\Omega_{\perp}$ is such that at (u,v),

$$\Omega_{\perp}(\sqrt{u^2 + v^2}\ e^{-\arctan(v/u)}) = - \Omega(\sqrt{u^2 + v^2}\ e^{\arctan(u/v)}), \tag{39}$$

then (38) and (39) do in fact determine curves that interest orthogonally.

The parametric form of the invariant differential equations (37) and (38) is as follows:

$$\frac{d}{d\sigma}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Omega & 1 \\ -1 & -\Omega \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$ (40)

and

$$\frac{d}{d\sigma}\begin{pmatrix} u_\perp \\ v_\perp \end{pmatrix} = \begin{pmatrix} -1 & -\Omega_\perp \\ \Omega_\perp & -1 \end{pmatrix}\begin{pmatrix} u_\perp \\ v_\perp \end{pmatrix}.$$ (40$_\perp$)

The corresponding second order systems are then

$$\left(\frac{d^2}{d\sigma^2} + \Omega\frac{d}{d\sigma} + [1 + \frac{d\Omega}{d\sigma}]\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$ (41)

or, introducing the expressions for $\frac{du}{d\sigma}$ and $\frac{dv}{d\sigma}$ from (40),

$$\left(\frac{d^2}{d\sigma^2} + [1 + \frac{d\Omega}{d\sigma} - \Omega^2]\right)\begin{pmatrix} u \\ v \end{pmatrix} = 2\Omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$ (42)

and

$$\left(\frac{d^2}{d\sigma^2} - [1 - \Omega_\perp^2]\right)\begin{pmatrix} u_\perp \\ v_\perp \end{pmatrix} = \left(2\Omega_\perp - \frac{d\Omega_\perp}{d\sigma}\right)\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_\perp \\ v_\perp \end{pmatrix}.$$ (41$_\perp$)

The form (41) is best suited for analyzing the microtime behavior of the (u,v)-field. Expressions (41$_\perp$) and (42) display the coupled nonlinear oscillatory behavior characteristic of the visual integrative process. Recalling that the invariant argument of Ω is essentially of the form $\sqrt{u^2 + v^2}\exp\{-\arctan(v/u)\}$, we see that (41$_\perp$) and (42) can be thought of as a sort of vector Lienard equation in which each of the field variables acts as forcing function for the other.

V. VISUAL PHENOMENA ASSOCIATED WITH THE DIFFERENTIAL EQUATIONS OF THE VISUAL INTEGRATIVE PROCESS

1. Images Evoked Under Flicker Stimulation

A standard procedure during the taking of an electroencephalograph is the illumination of the closed eyes of the subject by periodic flashes of light at about the frequency of the alpha rhythm. Under such circumstances a whirling spiral image is often evoked, and "whirlpools, explosions, Catherine wheels" are also seen (Walter, 1963, p. 101).

It is known that a sudden flash of light into the eyes will stimulate the brain into damped electrical oscillations whose frequency is approximately that of the alpha rhythm (Wooldridge, 1963, p. 109). However, the repetition of such light pulses at a frequency near the critical flicker-fusion frequency apparently results in a constant stimulus, as it appears to the visual cortex. In such circumstances the visual differential equations (37) and (38) provide a direct explanation of these evoked phenomena in the following way:

Let Ω = const. in (37), say, i.e., regard the stimulus as persisting over the whole visual field. Then (37) can be written in polar coordinates as

$$\frac{d \ln r}{d\theta} = + \Omega. \tag{43}$$

The solution of this equation is the logarithmic spiral

$$r = r_0 e^{+\Omega\theta}. \tag{44}$$

The companion differential equation (38) also yields a logarithmic spiral

$$r_\perp = r_{0\perp} e^{-9_\perp/\Omega_\perp}.$$ (45)

These expressions thus describe the spiral images evoked in the visual field by a stimulus that is "constant", i.e., the same, in the limit of time resolution, over the whole family of Ω-contours. The whirling character of the spiral follows from letting σ denote time in the parametric form (40) of the visual equations, and taking into account the resulting time variation. The other types of evoked images are more characteristic of the rest state of the visual cortex, which we now proceed to discuss.

2. MacKay's Complementary After Images

MacKay (1961), in his researches on the perception of regular, spatially repetitive patterns, found that long range interactions between widely separated portions of the visual field were apparently essential to the visual integrative process. He noted four types of striking visual effects evoked by repeated patterns:

(i) "Moiré" effects, which appear as flickering shadows in such patterns as that of fig. 8. They disappear upon retinal stabilization, and thus are apparently excited by the normal eye scanning movements.

(ii) The "subjective" colors observed in line patterns in motion by Erb and Dallenbach (1939).

(iii)A "streaming phenomenon" drifting at right angles to the lines in a pattern.

(iv) The complementary after image (CAI), consisting of a shimmering
after effect seen for a few seconds on a blank background as a scurry of
wavy lines or shadows moving in directions essentially perpendicular to
the lines in the stimulus pattern (figs. 8,9,10). For details of the
experimental procedures the reader is referred to the original paper
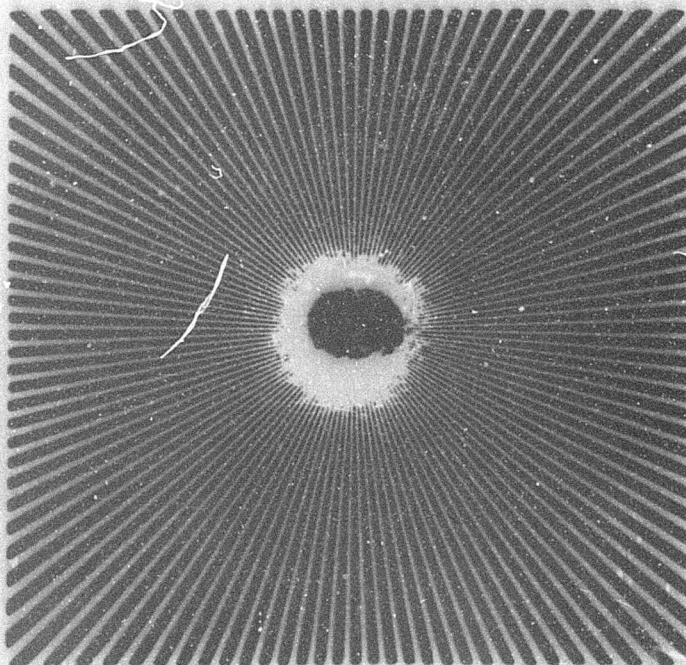(MacKay, 1961).

Fig. 8 A radial pattern that induces a circularly symmetric
complementary after image. (After MacKay (1961)).

MacKay established that neither (iii) nor (iv) are related to eye
movements. We note that figs. 8 and 9 taken together are essentially
the same as fig. 11 below, describing the cortical rest state. MacKay's
fig. 4 (reproduced here as fig. 10) shows the oscillatory perturbations
imposed on a family of circles after stimulation by the radial line
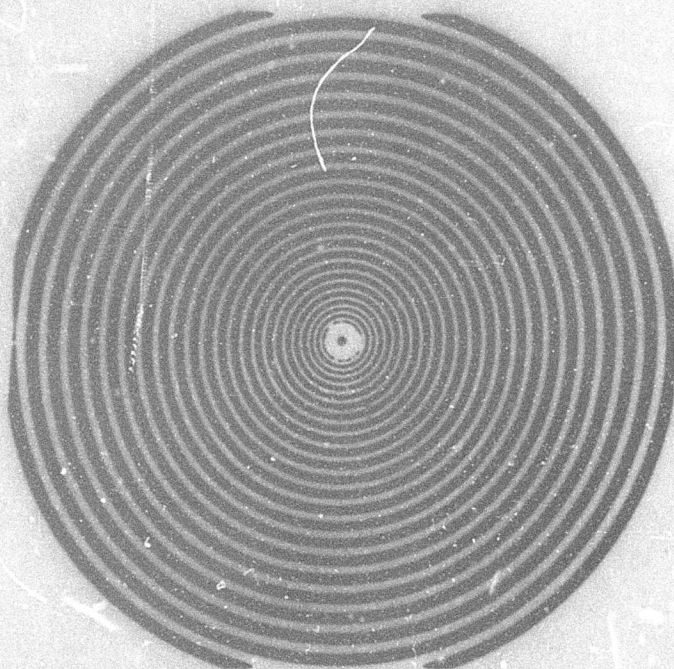pattern of fig. 8.

Fig. 9 A circular pattern that induces a radial complementary after image. (After MacKay (1961)).
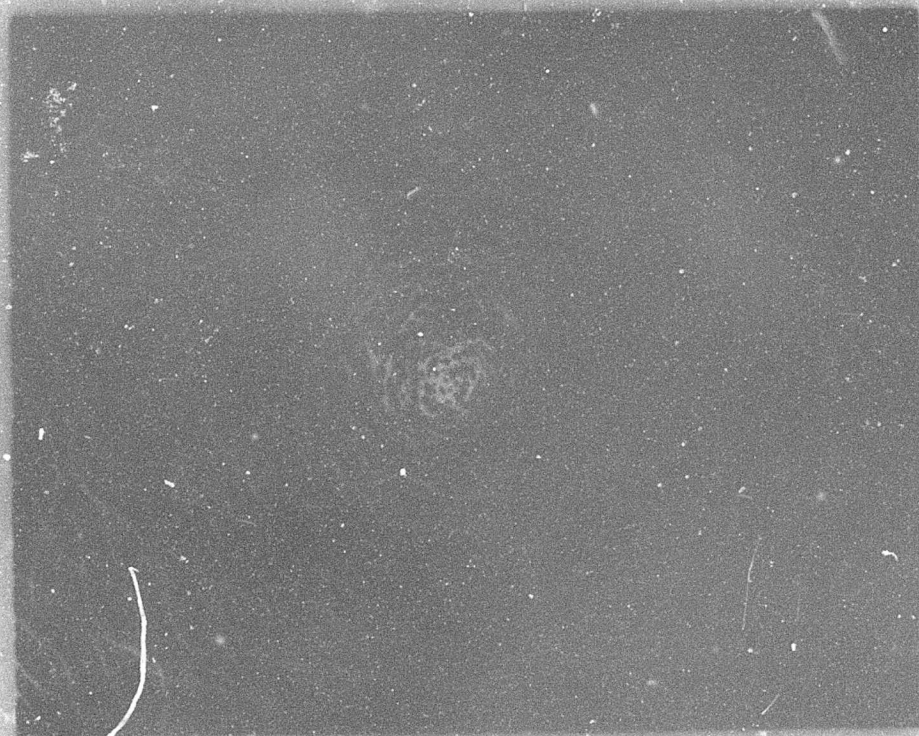

Fig. 10. An impression of a complementary after image induced by fig. 8. (After MacKay (1961)).

We now note that the cortical rest state (no visual image present) may be described (neglecting spontaneous discharges) by (37) and (38) with $\Omega = \Omega_{\perp} = 0$, i.e.,

$$\frac{dv}{du} = -\frac{u}{v} \qquad \text{and} \qquad \frac{dv_{\perp}}{du_{\perp}} = \frac{v_{\perp}}{u_{\perp}} . \tag{46}$$

The corresponding second order parametric equations are, from (42),

$$\frac{d^2u}{d\sigma^2} + u = 0, \qquad\qquad \frac{d^2v}{d\sigma^2} + v = 0 \tag{47}$$

$$\frac{d^2u_{\perp}}{d\sigma^2} - u_{\perp} = 0, \qquad\qquad \frac{d^2v_{\perp}}{d\sigma^2} - v_{\perp} = 0. \tag{48}$$

The pair (47) represent true sinusoidal oscillations in u and v; (48) on the other hand represent damped oscillations. It thus appears that sustained oscillations are possible only transverse to the u,v-coordinates of the visual field. We shall have more to say in this connection in our discussion below of the alpha rhythm.

The integrals of (46) are well known:

$$u^2 + v^2 = a^2 \qquad \text{and} \qquad v_{\perp} = a_{\perp}u_{\perp}. \tag{49}$$

The first represents a family of circles concentric about the origin; the second an orthogonal family of radial lines (fig. 11). We note that the intersection of a given radial line and a given circle is enough to define a point in the visual field unambiguously.

It thus appears from MacKay's work and the above theoretical result that stimulation of the visual cortex by a repetitive, radially symmetric pattern that essentially imposes a resonance on the cortical rest pattern (as exemplified by the curves (49) and fig. 11) can bring about a "ringing"

of the visual response in directions conjugate to the stimulus pattern.
We note in this connection that some of MacKay's subjects that suffered
from astigmatism had difficulty in seeing CAI (MacKay, 1961, p. 346).
Such a situation could come about through a failure to achieve resonance
between stimulus pattern and an asymmetric cortical rest state resulting
from continued astigmatic distortion. The usual cortical rest state,
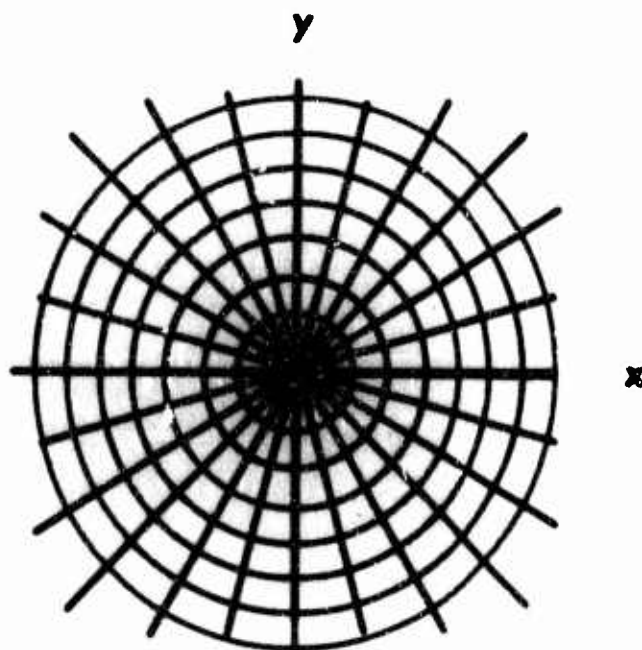as indicated by (49) and fig. 11, is inherently symmetric.

Fig. 11 Integral curves for the differential equations of
the cortical rest state. Note the resemblance to
the patterns in figs. 8 and 9.

The visual phenomena of the cortical rest state outlined above
also apparently bear directly upon the circular images in the eigengrau
described by Jung (Jung, 1961a, p. 295).

2. Developmental Dyslexia as a Variant of the Visual Differential Equations

Developmental dyslexia is a form of specific language disability that appears in childhood as an almost crippling inability to read and/or write (Bender, 1957; Money, 1961; Stuart, 1963). It is characterized primarily by an inordinately large number of confusions of letter symbols, especially reversals ("b" for "d", "p" for "q", "saw" for "was", etc.), which persists well past the first year of school. The disability is neurologically intrinsic: No brain damage or significant impairment of vision or hearing are present, and the only emotional disturbance is that attributable to continued classroom difficulties. Since these children (about one out of every ten) are unable to learn by the "sight method", they are in a very real sense victims of our standard educational system. The situation is doubly unfortunate since these children seem to be extra well endowed with creativeness and the capacity for abstract thought (Bender, 1957).

A full treatment of developmental dyslexia from the standpoint of visual differential equations may be found in Part II of this study. Here we content ourselves with outlining only those aspects of the visual differential equations that appear to be involved in the reversals described above.

If the sign of the slope is changed in (46) we obtain the new system

$$\frac{dv'}{du'} = \frac{u'}{v'}, \qquad \text{and} \quad \frac{dv'_\perp}{du_\perp'} = -\frac{v'_\perp}{u'_\perp}. \tag{50}$$
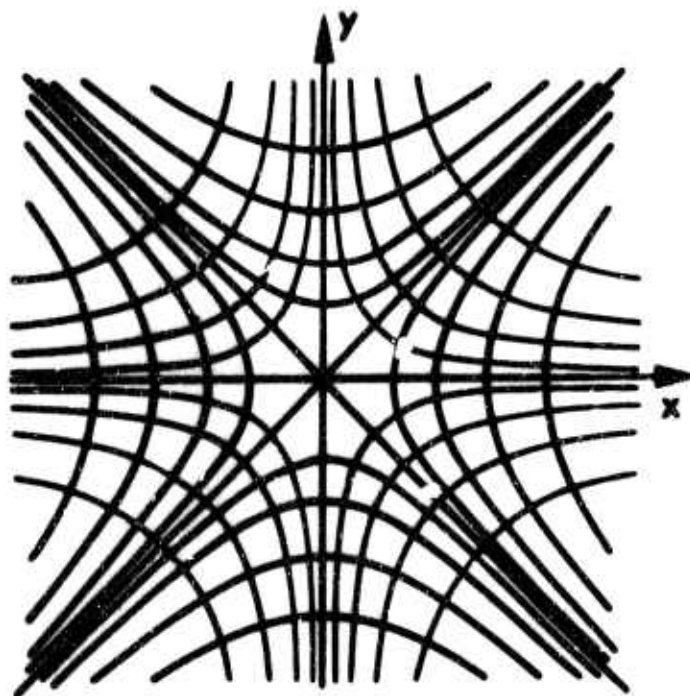
Fig. 12.   Integral curves for the rest-state differential equations
with reversed slope: conjugate families of hyperbolas.

The integral curves for this system are the conjugate families of hyper-
bolas (fig. 12):

$$v'^2 - u'^2 = a'^2, \quad \text{and} \quad u'_\perp v'_\perp = a'_\perp. \tag{51}$$

Eqs. (50) may be looked upon (see Part II) as the rest state equations
for a visual Gestalt whose basic invariances are with respect to (clock-
wise) rotations, magnification, and glide displacements. The essential
difference between systems (46) and (50) is that hyperbolic, rather than
radial, displacements are involved.

A similar, no doubt closely related hyperbolic frame of reference
arises in a natural way in Luneburg's theory of binocular vision.
The hyperbolic visual metric is probably the first to develop

in the infantile visual cortex. Evidence for the existence of such a
visual system during infancy can be gleaned from a study of Gesell, Ilg,
and Bullis (1950). Sometime during the early years of life - commonly
no later than the first year of school - system (46) becomes dominant.
The alpha rhythm usually becomes well established at this time and the
dysrhythmia characteristic of the infantile (and dyslexic) electro-
encephalograph almost entirely disappears.

Fig. 13 is intended to indicate the symmetries in such a hyperbolic
reference system that could bring about the letter and word reversals
described above. The origin in system (46) is a point of neutral sta-
bility, a so called vortex (fig. 11). For the reversed system (50) the
origin constitutes a saddle point, a type of singularity that is always
unstable. This suggests that the foveal part of the visual field is
inherently unstable in the dyslexic case. As a consequence, for an object
near the center of vision, the visual integrative process could jump in
a seemingly random manner from one quadrant to another and from one branch
of the reference hyperbola to the other. The confusion that this would
induce in a reading situation is not hard to imagine. The peripheral
vision on the other hand should be relatively unaffected.

The second order equations that follow from the parametric form of
(50) are all of the damped oscillation type:

$$\frac{d^2}{d\sigma^2} \begin{pmatrix} u' \\ v' \\ u'_\perp \\ v'_\perp \end{pmatrix} - \begin{pmatrix} u' \\ v' \\ u'_\perp \\ v'_\perp \end{pmatrix} = 0.$$
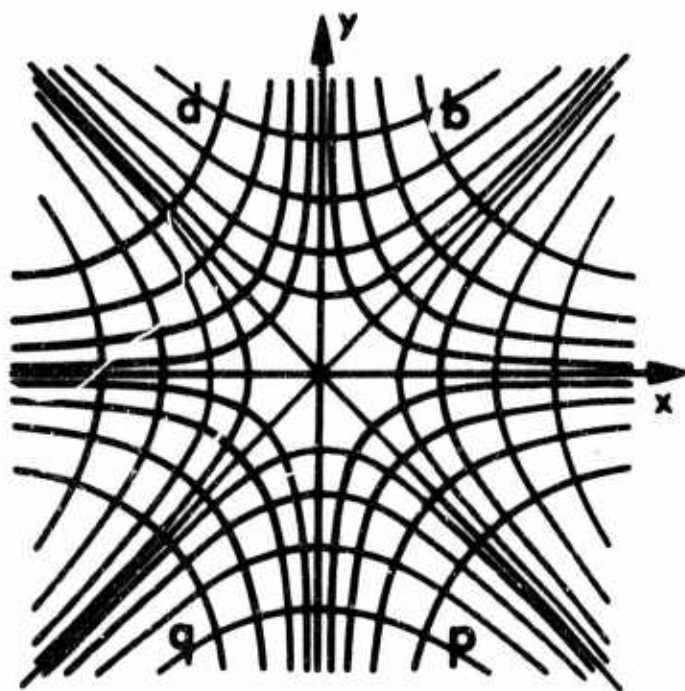
Fig. 13 Reproduction of fig. 12 with impressed letters "b","d",
"q", and "p". Each letter is symmetric with respect to
the horizontal or vertical axis separating it from the
next, and symmetric about the origin to the letter in
the opposite quadrant. The origin is a saddle point,
a position of unstable equilibrium.

This feature, too, is consistent with the immature dysrhythmic type of

electroencephalograph commonly encountered in dyslexic children (Bender,

1957;Cohn, 1949, p. 597).

This of course cannot be the whole story. The physiological origin

of the slope reversal is not clear, although nystagmus and labyrinthine

polarization may play a role (Jung, 1961a,b). Direction of cortical

scan, anomalous depth reversal of electrical polarization within the

cortex (Amassian, 1961), ontogeny*,learning, and biochemical variations

may also enter. Higher recognition phenomena must also involve the sort

of prediction and statistical decision processes that are described in

Section VI of this study. Even so, enough characteristic features of

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* Which traditionally recapitulates phylogeny.

developmental dyslexia fit the reversed slope model - which in itself
is based on known visual neurophysiology - to provide a working hypo-
thesis.

4. The Alpha Rhythm and Its Desynchronization

The alpha rhythm is a more or less regular oscillation, in the
frequency range 8 - 13 c.p.s., of the electric potential of the brain.
It is most prominent in the occipital region of the head, and is appar-
ently closely associated with the rest state of the visual cortex. The
alpha rhythm appears in the electroencephalograph only when the subject
is relaxed and resting, with eyes closed, and mind free from visualiza-
tion or deliberation. Mental problem solving or opening the eyes (even
in a darkened room) result in the "desynchronization" of the alpha
rhythm, i.e., its replacement by a low-amplitude oscillation of rela-
tively rapid fluctuation.

The frequency spectrum of the alpha rhythm is centered at about
10 c.p.s. (Wiener, 1958, p. 68). Other subsidiary peaks in the frequency
spectrum of the electrical activity of the brain occur at about 5 c.p.s
(the so called theta rhythm at 4 - 7 c.p.s.) and near d.c. (the so called
delta rhythm at 0.5 - 3.5 c.p.s.). The frequency range of the rapid
fluctuations characteristic of alert mental activity (the so called beta
rhythm) is 14 - 30 c.p.s.(Walter, 1953).

The ratios between these frequency ranges are strongly suggestive
of either subharmonic oscillations or parametric excitation (Minorsky,
1962, Chs. 19,20). The "desynchronization" of the alpha rhythm upon

presentation of a visual stimulus has much in common with such well known differential equation phenomena as asynchronous quenching of a subharmonic oscillation (Minorsky, 1962, Ch. 24) and the detuning of a parametric resonance by a sufficiently large frequency excursion (Minorsky, 1962, sec. 20.5).

Although subharmonic and parametric resonance are essentially different in origin, they yield phenomena which are in appearance indistinguishable. In true subharmonic resonance the energy source is external and so appears by itself on the right hand side of the governing differential equation as a periodic forcing function. In parametric resonance on the other hand the excitation arises through a periodic variation of some coefficient (or coefficients) of a homogeneous differential equation. The essential subharmonic nature, however, is present in both types of excitation. In both subharmonic and parametric oscillations the amplitude of the resonance oscillation decreases with increasing detuning and eventually disappears at a certain critical level.

An inspection of the invariant differential equations (37) and (38) indicates that it is more with an unusual form of parametric excitation than with asynchronous quenching of a subharmonic resonance that we have to do in the present instance. The image excitation appears as a coefficient of the form variables $(u,v),(u_\perp,v_\perp)$ rather than by itself as a driving term.

If we consider the microtime behavior of a single neuron, it displays a characteristic repetitive firing in response to a stimulus (Jung, 1961a,b). Although these neuronal discharges are certainly not strictly periodic, they do constitute "recurrent ...otions" in the terminology of the qualitative theory of differential equations (Nemytskii and Stepanov, 1960) and even appear to be almost periodic (ibid). The characteristic behavior of the neuron at (u_0, v_0) in the visual field may thus be represented as

$$\Omega\left(\sqrt{u^2 + v^2}\ e^{\arctan(u/v)}\right)\Bigg|_{(u_0, v_0)} = J(u_0, v_0) \sum_{\tau_\varkappa \leq t} \delta(t - \tau_\varkappa), \qquad (52)$$

where J is a coefficient characteristic of the particular type $(A, B, C, D, \text{ or } E)$ of neuron activated (Jung, 1961a,b), $\delta(t - \tau)$ denotes the delta function, and the $\tau_\varkappa$'s comprise a relatively dense set (Nemytskii and Stepanov, 1960) of time intervals characteristic of the neuronal type involved. $J(u_0, v_0)$ vanishes if the neuron at (u_0, v_0) is not activated by the visual stimulus.

It is clear from the second order visual field equations (41) with (52) for the image function Ω, that it is a radically different form of the differential equation of parametric excitation – if indeed it can be called that at all – that is involved. In fact the individual equations resemble more, perhaps, a sort of generalized Lienard equation with generalized functions as coefficients. Although conditions for the existence of a periodic solution of the Lienard equation are well known for continuous coefficients and periodic forcing function (Cesari, 1963), nothing is apparently known about the badly discontinuous case exemplified by combining (41) and (52).

To obtain some idea of the behavior of u and v under such circumstances, let us transpose all terms containing Ω in (41) to the right hand side and consider the equivalent integral equation involving generalized functions (Kaplan, 1962):

$$u(t) = K_1 \sin t + K_2 \cos t - J(u_0,v_0) \sum_{\{x\}} \int_0^t \sin(t - t')[\delta(t' - \tau_x)(\frac{du_0}{dt'} + v_0) +$$

$$+ \delta'(t' - \tau_x)u_0]dt', \qquad (53)$$

$$v(t) = K_1' \sin t + K_2' \cos t - J(u_0,v_0) \sum_{\{x\}} \int_0^t \sin(t - t')[\delta(t' - \tau_x)(u_0 - \frac{dv_0}{dt'}) -$$

$$- \delta'(t' - \tau_x)v_0]dt', \qquad (54)$$

where $\{x\}$ denotes the set of $\tau_x \leq t$. Evaluation of the integrals in (53) and (54) by the usual formulas of the theory of generalized functions leads at once to the forms

$$u(t) = \sin t \left\{ K_1 - J \sum_{\{x\}} [u_0(\tau_x)\sin \tau_x + v_0(\tau_x)\cos \tau_x] \right\} +$$

$$+ \cos t \left\{ K_2 - J \sum_{\{x\}} [u_0(\tau_x)\cos \tau_x - v_0(\tau_x)\sin \tau_x] \right\}, \qquad (55)$$

and

$$v(t) = \sin t \left\{ K_1' + J \sum_{\{x\}} [u_0(\tau_x)\cos \tau_x - v_0(\tau_x)\sin \tau_x] \right\} +$$

$$+ \cos t \left\{ K_2' - J \sum_{\{x\}} [u_0(\tau_x)\sin \tau_x + v_0(\tau_x)\cos \tau_x] \right\}. \qquad (56)$$

If these expressions are to satisfy (40) also, then we must have

$$K_1' = - K_2, \qquad K_2' = K_1, \qquad (57)$$

and the following system must hold as well:

$$K_1 \sum_{\{\mu\}} \sin \tau_\mu + K_2 \sum_{\{\mu\}} \cos \tau_\mu = J \sum_{\{x\}} \left\{ u_0(\tau_x) \sum_{\{\mu\}} \cos(\tau_x - \tau_\mu) - v_0(\tau_x) \sum_{\{\mu\}} \sin(\tau_x - \tau_\mu) \right\}$$

(58)

$$-K_1 \sum_{\{\mu\}} \cos \tau_\mu + K_2 \sum_{\{\mu\}} \sin \tau_\mu = - J \sum_{\{x\}} \left\{ u_0(\tau_x) \sum_{\{\mu\}} \sin(\tau_x - \tau_\mu) - v_0(\tau_x) \sum_{\{\mu\}} \cos(\tau_x - \tau_\mu) \right\}.$$

Solving the system (58) and putting the result and (57) back into (55) and (56), one finds, after some straightforward trigonometric reductions, the almost periodic solutions

$$u(t) = \frac{J'}{\sum_{\{\mu,\nu\}} \frac{1}{\mu - \tau_\nu}} \left(\sum_{\{x\}} u_0(\tau_x) \sum_{\{\mu,\nu\}} \cos(t - \tau_x + \tau_\mu - \tau_\nu) - v_0(\tau_x) \sum_{\{\mu,\nu\}} \sin(t + \tau_x - \tau_\mu - \tau_\nu) \right] -$$

(59)

$$- \sum_{\{\mu,\nu\}} \cos(\tau_\mu - \tau_\nu) \sum \left[u_0(\tau_x) \cos(t - \tau_x) + v_0(\tau_x) \sin(t - \tau_x) \right] \Big) ,$$

and

$$v(t) = \frac{J(u_0, v_0)}{\sum_{\{\mu,\nu\}} \cos(\tau_\mu - \tau_\nu)} \left(-\sum_{\{x\}} \left[u_0(\tau_x) \sum_{\{\mu,\nu\}} \sin(t - \tau_x + \tau_\mu - \tau_\nu) + v_0(\tau_x) \sum_{\{\mu,\nu\}} \cos(t + \tau_x - \tau_\mu - \tau_\nu) \right] + \right.$$

(60)

$$\left. + \sum_{\{\mu,\nu\}} \cos(\tau_\mu - \tau_\nu) \sum_{\{x\}} \left[u_0(\tau_x) \sin(t - \tau_x) - v_0(\tau_x) \cos(t - \tau_x) \right] \right).$$

The almost periodic character of the u,v-oscillations excited by a neuron at (u_0, v_0) firing at times $\tau_1, \tau_2, \ldots, \tau_N \ (\leq t)$ is evident from inspection of (59) and (60). If the neuron is of Jung's type A, which discharges more or less regularly without a specific stimulus, at a frequency in the alpha range (8 - 15 c.p.s.), then the oscillations determined by (59) and (60) will have the observed characteristics of the alpha rhythm. If a visual stimulus is presented, on the other hand, so

that the $\{\tau_m\}$ sequence is that of the more rapid and irregular dis-charges characteristic of neuron types B and D, and C and E, then a "desynchronization" of the A-type oscillation would appear. If we now suppose that the stimulus point (u_0, v_0) ranges over all the cortical neurons that are activated in the visual field (i.e., J now represents a summation over these (u_0, v_0)), then it is clear that we have an almost periodic oscillation with a distributed excitation. Qualitative versions of this result have been given by Lashley and Sholl (Sholl, 1956).

We should not lose sight of the fact that the relatively dense set $(\tau_1, \ldots, \tau_N)$ is essentially a random sequence as well. It is therefore natural to associate an oscillatory random process (Wold, 1954, p. 15) with it. The more or less regular oscillation of the alpha rhythm may then be viewed as an instance of Slutzky and Romanovsky's "sinusoidal limit theorem" (Slutsky, 1937), the burden of which is that even with a purely random series, repeated pairwise moving averages and m^{th} order differencing will generate a cyclic variation, which in the limit is sinusoidal.

VI. STATISTICAL DECISION AND PREDICTION THEORY

The Lie group operations that we have considered, and the invariant
differential equations (37) and (38) that follow from them, have involved
only very basic aspects of the visual integrative process. In actuality,
however, visual pattern recognition seems to be a stagewise process in
which the mind goes through a rapid series of comparisons with what it
already knows. Eventually, in proper Gestalt fashion, things click into
place and we recognize the pattern for what it is. Learning can often
shorten this puzzling over a given pattern quite considerably, which
again indicates the prominent role of memory and prediction in the recog-
nition process.

Such considerations as these lead us at once into the domain of
statistical decision and prediction theory. Not a great deal is known
in this connection for the sort of functional equations that are involved
in the visual integrative process. Research on the statistical aspects
of Lie groups is only little more than a decade old (Brillinger, 1963;
Grenander, 1963), and more work in this field is needed before a pre-
diction and decision theory fully adequate for our purposes can be
formulated. Brillinger's main theorem (Brillinger, 1963, p. 495) seems
to be the closest available result and would apparently suffice for the
basic aspects of the visual Gestalt if suitable decision procedures and
risk functions could be prescribed. This approach will be discussed in
detail in a later part of this study devoted specifically to memory and
statistical prediction and decision theory.

With regard to statistical invariance in general and the prediction theory for linear systems of random differential equations, we are in a somewhat better position (Bartlett, 1955; Beutler, 1963; Doob, 1953; Lehmann, 1959; Shiryaev, 1963; Wesler, 1959). The restricted minimax and modified minimax (or "slicing") principles (Hodges and Lehmann, 1952; Wesler, 1959) are each directed toward making maximum use of previous experience in a statistical decision procedure, and thus appear especially appropriate in the present context. Wesler's approach, via partitioning the set of states of nature (images) into subclasses or "slices", exhibits the stagewise sort of decision procedure that appears to be involved in visual pattern recognition.

The visual differential equations (40) and (40$_1$), with memory, would seem to be a vector, predictive form of the stochastic differential equation studied by Ito (1951) and others. Systems whose parameters vary randomly in time, i.e., random parametric excitation, have also received study (see Bogdanoff and Kozin (1963), or Caughey and Dienes (1962), and the references given there), but no prediction theory appears to have been formulated. Some parts of the theory of random parametric excitation are still controversial, and much remains to be done here also. The random differential equation aspect, as well as that of the modified minimax decision principle briefly mentioned in the preceding paragraph, will also be taken up in a later part of this study devoted to statistical questions.

The introduction of statistical decision and prediction theory automatically raises the matter of memory. The relative roles of bio-molecular (Hydén, 1960; Hydén and Egyházi, 1962, 1963) and neuronal

(Dingman and Sporn, 1964) mechanisms in animal memory is immaterial to our memory model. Changes in the travel time of the electrical nerve impulse through the neural net arising out of either local biochemical modification of the neuronal electrical conductivity or dendritic lengthening through axoplasmic flow could account for memory phenomena in terms of oscillations in systems with time retardations (Minorsky, 1962, Ch. 21). Even though the number of time lags is finite, the difference-differential equations that are involved in such retardation phenomena always have an infinite spectrum, i.e., such a system can in principle oscillate at an infinite number of frequencies, thus providing unlimited memory capacity. In practice the retardations cause damping of the oscillations, so that only a finite number of components contribute significantly to any given oscillation. In many respects these retarded oscillations resemble autoregressive schemes (Wold, 1954), and the phenomenon of parametric excitation, which has been discussed briefly above and in sec. V.4, is also involved.

VII. CONCLUSION

The material covered in this report divides naturally into three categories: (i) the cortical counterpart of the method of isoclines, the visual Gestalt in terms of Lie group operations, and the invariant differential equations of the visual integrative process; (ii) comparison with experimental evidence: flicker phenomena, complementary after-images, developmental dyslexia as slope reversal in the visual differential equations, and the alpha rhythm and its "desynchronization"; (iii) memory and recognition in terms of statistical decision and prediction theory. Only (i) has been given a full treatment here. Detailed expositions of the role of slope reversal in developmental dyslexia and the statistical decision and prediction theory required for a fuller explanation of Gestalt and memory phenomena will be reserved for separate publications.

Two other important aspects of visual pattern recognition have not been discussed in this part (I) of the study, viz., the cortical differential processes involved in the recognition of motion and curvature and the neuronal interconnections required by the isocline model and the visual differential equations (37) and (38). The former is easily handled in terms of the well known calculus formulas:

$$V_\xi = \frac{d\xi}{dt} = \dot{\sigma}\, \frac{d\xi}{d\sigma} \; ; \tag{61}$$

$$a_\xi = \frac{d^2\xi}{dt^2} = \dot{\sigma}^2\, \frac{d^2\xi}{d\sigma^2} + \ddot{\sigma}\, \frac{d\xi}{d\sigma} \; ; \tag{62}$$

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while, for fixed time, the curvature is given by

$$\varkappa(s) = [(\frac{d^2u}{ds^2})^2 + (\frac{d^2v}{ds^2})^2]^{1/2}; \tag{63}$$

or

$$\varkappa(\sigma) = (\frac{du}{d\sigma}\frac{d^2v}{d\sigma^2} - \frac{dv}{d\sigma}\frac{d^2u}{d\sigma^2})/[(\frac{du}{d\sigma})^2 + (\frac{dv}{d\sigma})^2]^{3/2}. \tag{64}$$

In these expressions ζ denotes either u or v, V_ζ is the velocity in the ζ direction, t is time (a dot over a variable indicates differentiation with respect to time), σ is the cortical parameter (phase, time, or whatever), a_ζ is the acceleration in the ζ direction, $\varkappa$ denotes local curvature, and s represents arc length.

Whether the simple fields are actually interconnected within the cortex in such fashion as to process first and second derivatives according to the prescriptions in (61)-(64) is an open question. However, many investigators have noted the necessity for something that is at least equivalent to these operations. As Granit (1962, p. 756) puts it, "... the cortical response ... is adjusted so as to emphasize everything that involves discrimination, i.e., small fields, direction, movement, contrast, almost as if it were taking the second derivative ...". We further noted in our discussion of Hubel and Wiesel's work (sec. II) that, while both simple and complex fields were sensitive to motion of the light stimulus, the complex fields were especially so. This suggests that the simple fields are interconnected within the complex fields in such a way as to estimate (61) and (62). Such cortical computation could be accomplished either by further reciprocally antagonistic inhibition of the first differences given by the simple fields

or by a further application of the method of isoclines (Cunningham, 1958, sec. 3.3) to differential equations like (42) and $(41_\perp)$.

Concerning the neuronal interconnections involved in the simple fields that enable the cortex to solve (40) and $(40_\perp)$ approximately, there does not appear to be enough neurophysiological data of the right type to permit precise statements. A start in this direction has been made by Gerstein and Clark (1964), but even then their work is not oriented toward the differential properties of the neuronal interactions. The history of cytoarchitectural studies of the cortical neurons is well set forth by Sholl (1956), but this bears much more on structure than on function[*].

The inhibitory character, both reciprocally antagonistic (time) and lateral (space), of the neuronal action suggests that first and higher differences are bound to play an important role in the network of neuronal interconnections. The work of Fessard (1961) is of considerable interest in this regard. We further note that such difference operations are not inconsistent with the autocorrelation principle advanced by Reichardt (1961), since the latter is implicit in such difference operations as

$$\text{average}(\xi_1 - \xi_2)^2 = \text{variance}(\xi_1) + \text{variance}(\xi_2) - 2\,\text{covariance}(\xi_1, \xi_2),$$

whenever $\text{average}(\xi_1) = \text{average}(\xi_2)$.

----

[*] Even so, the resemblance between the structural character of the principal types of neuron present in the cerebral cortex (Sholl, 1956) and the three types of components (i.e., sets of trajectories) that are possible in such systems as (40) or $(40_\perp)$ if they are to be structurally stable (De Baggis, 1952) is worthy of note.

This concludes Part I of the study. Subsequent parts will deal in detail with slope reversal in the visual differential equations and developmental dyslexia, statistical decision and prediction theory for visual memory and Gestalt, and the information theoretic aspects of pattern recognition by the method of isoclines.

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